



2.3.3 Properties of vector operations





Introduction

- In this topic, we will
 - Define vector spaces
 - Look at some properties of vector spaces
 - List the vector space axioms
 - See how other properties can be deduced from these axioms





Vector spaces

- For each positive integer n ,
 - We will define $\mathbf{R}^n$ together with the operations of scalar multiplication and vector addition as a real *vector space*
 - We will define $\mathbf{C}^n$ together with the operations of scalar multiplication and vector addition as a complex *vector space*
- For example, we may refer to the vector space of all 3-dimensional real vectors
 - The operations of scalar multiplication and vector addition as we have defined them are assumed





Vector spaces

- Given a vector space $\mathbf{F}^n$,
given m vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ all in $\mathbf{F}^n$, and
 m scalars $\alpha_1, \dots, \alpha_m$ all in $\mathbf{F}$
we will call

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m$$

a *linear combination* of these m vectors and corresponding scalars

- For example,

$$2.3 \begin{pmatrix} 8 \\ 5 \\ 6.5 \\ 2.3 \end{pmatrix} - 4.7 \begin{pmatrix} 4.6 \\ 5.7 \\ 0.3 \\ 2.9 \end{pmatrix} + 5.9 \begin{pmatrix} 2.2 \\ 9.9 \\ 7.3 \\ 4.4 \end{pmatrix} - 8.6 \begin{pmatrix} 0.2 \\ 4.8 \\ 8.8 \\ 9.8 \end{pmatrix} + 0.1 \begin{pmatrix} 3.8 \\ 1.7 \\ 4.6 \\ 1.9 \end{pmatrix} = \begin{pmatrix} 8.42 \\ 2.01 \\ -18.61 \\ -66.47 \end{pmatrix}$$





Properties

- There are lots of useful properties of vector spaces:
 1. Given any n vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ and any m scalars $\alpha_1, \dots, \alpha_m$,
we can add $\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m$ in any order we want
 2. There is a unique zero vector $\mathbf{0}_n$ and $0\mathbf{v} = \mathbf{0}_n$ for all vectors $\mathbf{v}$,
 $\alpha \mathbf{0}_n = \mathbf{0}_n$ for all scalars α , and if $\mathbf{u} + \mathbf{v} = \mathbf{u}$, then $\mathbf{v} = \mathbf{0}_n$
 3. $1 \cdot \mathbf{u} = \mathbf{u}$ and if $\alpha \mathbf{u} = \mathbf{u}$, then $\alpha = 1$ or $\mathbf{u} = \mathbf{0}_n$ or both
 4. Every $\mathbf{u}$ has a unique additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}_n$,
 $-\mathbf{u} = -1 \cdot \mathbf{u}$, $-(-\mathbf{u}) = \mathbf{u}$, $-\mathbf{0}_n = \mathbf{0}_n$ and given the m vectors and
scalars above above,
$$-(\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m) = \alpha_1 (-\mathbf{v}_1) + \dots + \alpha_m (-\mathbf{v}_m)$$
and if $\mathbf{u} + \mathbf{v} = \mathbf{0}_n$, then $\mathbf{v} = -\mathbf{u}$
 5. Also, $\alpha \mathbf{u} = \mathbf{0}_n$ if and only if either $\alpha = 0$, $\mathbf{u} = \mathbf{0}_n$ or both
 6. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$





Properties

- Also, if we define $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$, then
 1. $\mathbf{u} + \mathbf{v} = \mathbf{w}$ if and only if $\mathbf{w} - \mathbf{u} = \mathbf{v}$ and $\mathbf{w} - \mathbf{v} = \mathbf{u}$
 2. $\alpha(\mathbf{u} - \mathbf{v}) = \alpha\mathbf{u} - \alpha\mathbf{v}$
 3. $-(\mathbf{u} - \mathbf{v}) = \mathbf{v} - \mathbf{u}$
 4. $(\alpha - \beta)\mathbf{u} = \alpha\mathbf{u} - \beta\mathbf{u}$





Vector space properties

- Recall that we saw that the real numbers had a number of nice properties, most of which you were already familiar with
 - We asked: what is a minimal set of properties that are required so that all the nice properties still hold?
 - The answer was the eight field axioms
 - We found the complex numbers satisfy these axioms ☺
- We now have have to ask: what is a minimal set of properties that are required so that all the nice properties of real and complex finite-dimensional vectors still hold?
 - The answer is a set of ten vector space axioms
 - We will find many other “vector spaces” that look nothing like our finite dimensional vector spaces $\mathbf{R}^n$ and $\mathbf{C}^n$





Vector space axioms

- Scalar multiplication axioms
 1. If $\mathbf{u}$ is any vector, then $\alpha\mathbf{u}$ is also a vector for all scalars α
 2. $1 \cdot \mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u}$
 3. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ for all α, β in $\mathbf{F}$ and all vectors $\mathbf{u}$
- Vector addition axioms
 4. If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then $\mathbf{u} + \mathbf{v}$ is also a vector
 5. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all vectors $\mathbf{u}$ and $\mathbf{v}$
 6. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$
- The relationship between scalar multiplication and vector addition
 7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
 8. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
- The zero vector and additive inverses
 9. There is a vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all vectors $\mathbf{u}$
 10. For each vector $\mathbf{u}$, there is a vector $-\mathbf{u}$ such that $\mathbf{u} + -\mathbf{u} = \mathbf{0}$





Vector space axioms

- Any collections of “objects” associated with a field $\mathbf{F}$ where
 - Addition of these objects is defined
 - Scalar multiplication of these objects with scalars in $\mathbf{F}$ is defined
 - And all the ten axioms hold

we may call this collection a vector space, as well

- For such a vector space,
all the other nice properties of vectors also hold!





Vector space axioms

- We will look into these axioms, and then we will see how we can use them to deduce other properties





1. Closure under scalar multiplication

- The first is straight-forward:
 - We must be able to multiply any vector by any scalar in $\mathbf{F}$ and the result must still be a vector

- For example, if we consider only those 2-d real vectors

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where both $u_1 \geq 0$ and $u_2 \geq 0$, this cannot be a vector space

because $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is such a vector, and -1 is a real scalar, but

$$-1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$





2. A vector is unchanged if multiplied by 1

- The second is also true for real and complex vectors:

$$1 \cdot \mathbf{u} = \mathbf{u}$$

- For this property not to hold, we'd need a different definition of scalar multiplication

- Suppose we *defined* scalar multiplication on two-dimensional real vectors to be

$$\alpha \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} 2\alpha u_1 \\ 2\alpha u_2 \end{pmatrix}$$

- In this case, $1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$





3. Field multiplication associates with scalar multiplication

- The third axiom is

$$\alpha(\beta \mathbf{u}) = (\alpha\beta)\mathbf{u} \text{ for all } \alpha, \beta \text{ in } \mathbf{F} \text{ and all vectors } \mathbf{u}$$

- The third basically says to calculate $3 \cdot \left(4 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} \right)$

$$\text{you can calculate } 3 \cdot \left(4 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} \right) = 3 \cdot \begin{pmatrix} 20.8 \\ -6.4 \end{pmatrix} = \begin{pmatrix} 62.4 \\ -19.2 \end{pmatrix}$$

or you can do

$$3 \cdot \left(4 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} \right) = (3 \cdot 4) \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} = 12 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} = \begin{pmatrix} 62.4 \\ -19.2 \end{pmatrix}$$





4. Vector addition is closed

- The fourth axiom says that if you add two vectors, you get a vector back
- Let us restrict ourselves to vectors of the form $\begin{pmatrix} u_1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ u_2 \end{pmatrix}$
 - This includes only those vectors on the x - and y -axes
 - This set of vectors is closed under scalar multiplication:
 - If you multiply a vector on an axis by a scalar, it stays on that axis
 - This set, however, is not closed under vector addition:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$





5. Vector addition is commutative

- Because the entries of a vector are in a field, where addition is commutative, you sort-of expect vector addition to be commutative

$$\begin{pmatrix} 3.2 \\ -1.5 \end{pmatrix} + \begin{pmatrix} 4.7 \\ 8.9 \end{pmatrix} = \begin{pmatrix} 7.9 \\ 7.4 \end{pmatrix}$$

$$\begin{pmatrix} 4.7 \\ 8.9 \end{pmatrix} + \begin{pmatrix} 3.2 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 7.9 \\ 7.4 \end{pmatrix}$$





6. Vector addition is associative

- Because the entries of a vector are in a field, where addition is associative, you sort-of expect vector addition to be associative

$$\begin{pmatrix} 3.2 \\ -1.5 \end{pmatrix} + \left(\begin{pmatrix} 4.7 \\ 8.9 \end{pmatrix} + \begin{pmatrix} 5.0 \\ 6.1 \end{pmatrix} \right) = \begin{pmatrix} 3.2 \\ -1.5 \end{pmatrix} + \begin{pmatrix} 9.7 \\ 15.0 \end{pmatrix} \\ = \begin{pmatrix} 12.9 \\ 13.5 \end{pmatrix}$$

$$\left(\begin{pmatrix} 3.2 \\ -1.5 \end{pmatrix} + \begin{pmatrix} 4.7 \\ 8.9 \end{pmatrix} \right) + \begin{pmatrix} 5.0 \\ 6.1 \end{pmatrix} = \begin{pmatrix} 7.9 \\ 7.4 \end{pmatrix} + \begin{pmatrix} 5.0 \\ 6.1 \end{pmatrix} \\ = \begin{pmatrix} 12.9 \\ 13.5 \end{pmatrix}$$





7. Scalar multiplication distributes over vector addition

- The seventh axiom is

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \text{ for all } \alpha \text{ in } \mathbf{F} \text{ and all vectors } \mathbf{u} \text{ and } \mathbf{v}$$

- The seventh basically says to calculate $8 \cdot \left(\begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} + \begin{pmatrix} -4.9 \\ 3.7 \end{pmatrix} \right)$

$$\text{you can calculate } 8 \cdot \left(\begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} + \begin{pmatrix} -4.9 \\ 3.7 \end{pmatrix} \right) = 8 \cdot \begin{pmatrix} 0.3 \\ 2.1 \end{pmatrix} = \begin{pmatrix} 2.4 \\ 16.8 \end{pmatrix}$$

$$\begin{aligned} \text{or } 8 \cdot \left(\begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} + \begin{pmatrix} -4.9 \\ 3.7 \end{pmatrix} \right) &= 8 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} + 8 \cdot \begin{pmatrix} -4.9 \\ 3.7 \end{pmatrix} \\ &= \begin{pmatrix} 41.6 \\ -12.8 \end{pmatrix} + \begin{pmatrix} -39.2 \\ 29.6 \end{pmatrix} = \begin{pmatrix} 2.4 \\ 16.8 \end{pmatrix} \end{aligned}$$





8. Scalar multiplication distributes over field addition

- The eighth axiom is

$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u} \text{ for all } \alpha, \beta \text{ in } \mathbf{F} \text{ and all vectors } \mathbf{u}$$

- The eighth axiom basically says to calculate $(3 + 4) \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix}$

$$\text{you can calculate } (3 + 4) \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} = 7 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} = \begin{pmatrix} 36.4 \\ -11.2 \end{pmatrix}$$

$$\begin{aligned} \text{or } (3 + 4) \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} &= 3 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} + 4 \cdot \begin{pmatrix} 5.2 \\ -1.6 \end{pmatrix} \\ &= \begin{pmatrix} 15.6 \\ -4.8 \end{pmatrix} + \begin{pmatrix} 20.8 \\ -6.4 \end{pmatrix} = \begin{pmatrix} 36.4 \\ -11.2 \end{pmatrix} \end{aligned}$$





9. There is a zero vector

- The ninth axiom requires that there be zero vector $\mathbf{0}$ such that
$$\mathbf{0} + \mathbf{u} = \mathbf{u} \text{ for all vectors } \mathbf{u}$$
 - We've already discussed the zero vector





10. The existence of additive inverses

- The tenth axiom says that for each vector $\mathbf{u}$, there is a vector denoted $-\mathbf{u}$ such that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$





Additional properties

- All the other properties of vector spaces we listed can be deduced from these ten axioms

Theorem

The zero vector is unique.

Proof:

Suppose there are two vectors $\mathbf{0}$ and $\mathbf{z}$,

where $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $\mathbf{z} + \mathbf{u} = \mathbf{u}$ for all vectors $\mathbf{u}$.

Then $\mathbf{0} + \mathbf{z} = \mathbf{z}$ and $\mathbf{z} + \mathbf{0} = \mathbf{0}$,

but $\mathbf{0} + \mathbf{z} = \mathbf{z} + \mathbf{0}$ because vector addition commutes

Thus, $\mathbf{z} = \mathbf{0} + \mathbf{z} = \mathbf{z} + \mathbf{0} = \mathbf{0}$, but equality is transitive,
therefore $\mathbf{z} = \mathbf{0}$. ■





Additional properties

Theorem

$0 \cdot \mathbf{u} = \mathbf{0}$ for all vectors $\mathbf{u}$.

Proof:

Let $\mathbf{u}$ be any vector.

Therefore	$\mathbf{u} = 1 \cdot \mathbf{u}$	but $1 = 1 + 0$
	$= (1 + 0) \cdot \mathbf{u}$	but $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
	$= 1 \cdot \mathbf{u} + 0 \cdot \mathbf{u}$	but $1 \cdot \mathbf{u} = \mathbf{u}$
	$= \mathbf{u} + 0 \cdot \mathbf{u}$	but this says $\mathbf{u} = \mathbf{u} + 0 \cdot \mathbf{u}$

Therefore, $0 \cdot \mathbf{u}$ is a zero vector.

But the zero vector is unique (last theorem)

Therefore $0 \cdot \mathbf{u} = \mathbf{0}$. ■





Additional properties

Theorem

Given a vector $\mathbf{u}$, the additive inverse $-\mathbf{u}$ is unique.

Proof:

Let $\mathbf{u}$ be any vector and let $-\mathbf{u}$ be an additive inverse.

Suppose there is a second vector $\mathbf{v}$ such $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

Therefore $\mathbf{v} = \mathbf{v} + \mathbf{0}$

$$= \mathbf{v} + (\mathbf{u} + (-\mathbf{u}))$$

$$= (\mathbf{v} + \mathbf{u}) + (-\mathbf{u})$$

$$= (\mathbf{u} + \mathbf{v}) + (-\mathbf{u})$$

$$= \mathbf{0} + (-\mathbf{u})$$

$$= -\mathbf{u}$$

$$\text{but } \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

but by associativity

but by commutativity

$$\text{but } \mathbf{u} + \mathbf{v} = \mathbf{0}$$

As equality is transitive,

Therefore $\mathbf{v} = -\mathbf{u}$. ■





Additional properties

Theorem

$-\mathbf{u} = -1 \cdot \mathbf{u}$ for all vectors $\mathbf{u}$.

Proof:

Let $\mathbf{u}$ be any vector.

Therefore $\mathbf{0} = 0 \cdot \mathbf{u}$ but $1 + (-1) = 0$
 $= (1 + (-1)) \cdot \mathbf{u}$ but $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
 $= 1 \cdot \mathbf{u} + -1 \cdot \mathbf{u}$ but $1 \cdot \mathbf{u} = \mathbf{u}$
 $= \mathbf{u} + -1 \cdot \mathbf{u}$ but this says $\mathbf{0} = \mathbf{u} + -1 \cdot \mathbf{u}$

Therefore, $-1 \cdot \mathbf{u}$ is an additive inverse of $\mathbf{u}$.

But the additive inverse of a vector is unique (last theorem)

Therefore $-1 \cdot \mathbf{u} = -\mathbf{u}$. ■





Additional properties

Theorem

$\alpha \cdot \mathbf{0} = \mathbf{0}$ for all scalars α .

Proof:

Let α be any scalar in $\mathbf{F}$ and let $\mathbf{u}$ be any vector.

Therefore $\alpha \cdot \mathbf{0} = \alpha \cdot (0\mathbf{u})$

$$= (\alpha \cdot 0) \cdot \mathbf{u}$$

$$= 0 \cdot \mathbf{u}$$

$$= \mathbf{0}$$

but $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$

but $\alpha \cdot 0 = 0$ in any field

But equality is transitive.

Therefore $\alpha \cdot \mathbf{0} = \mathbf{0}$. ■





Additional properties

Theorem

$\alpha \mathbf{u} = \mathbf{0}$ if and only if either $\alpha = 0$, $\mathbf{u} = \mathbf{0}$ or both

Proof:

If $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$, $\alpha \mathbf{u} = \mathbf{0}$ from previous theorems

Assume $\alpha \neq 0$ and $\mathbf{u} \neq \mathbf{0}$, but $\alpha \mathbf{u} = \mathbf{0}$.

Because $\alpha \neq 0$, α has a multiplicative inverse α^{-1}

But $\alpha^{-1}\mathbf{0} = \mathbf{0}$ yet $\alpha^{-1}(\alpha\mathbf{u}) = (\alpha^{-1}\alpha) \mathbf{u} = 1 \cdot \mathbf{u} = \mathbf{u}$

By transitivity, this suggests $\mathbf{0} = \mathbf{u}$.

This contradicts our assumption.

Therefore, $\alpha \mathbf{u} \neq \mathbf{0}$ if neither $\alpha \neq 0$ and $\mathbf{u} \neq \mathbf{0}$. ■





Summary

- Following this topic, you now
 - Understand the vector space axioms
 - Understand that all additional properties can be derived from these axioms
 - You have seen a number of such proofs





References

- [1] https://en.wikipedia.org/wiki/Vector_space





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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